BILINEAR FORMS ON THE DIRICHLET SPACE

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1. Introduction

1.1. **Overview.** Let $\mathcal{D}$ be the classical Dirichlet space, the Hilbert space of holomorphic functions on the disk with inner product

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA$$

and normed by $\|f\|_{\mathcal{D}}^2 = \langle f, f\rangle_{\mathcal{D}}$. Given a holomorphic *symbol function* b we define the associated Hankel type bilinear form, initially for $f, g \in \mathcal{P}(\mathbb{D})$, the space of polynomials, by

$$T_b\left(f, g\right) := \langle fg, b\rangle_{\mathcal{D}}.$$

The norm of T_b is

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} := \sup\left\{\left|T_b\left(f, g\right)\right| : \|f\|_{\mathcal{D}} = \|g\|_{\mathcal{D}} = 1\right\}.$$

We say a positive measure μ on the disk is a *Carleson measure for* $\mathcal{D}$ if

$$\|\mu\|_{CM(\mathcal{D})} := \sup\left\{\int_{\mathbb{D}} |f|^2\, d\mu : \|f\|_{\mathcal{D}} = 1\right\} < \infty,$$

and that a function b is in the space $\mathcal{X}$ if the measure $d\mu_b := |b'(z)|^2\, dA$ is a Carleson measure. We norm $\mathcal{X}$ by

$$\|b\|_{\mathcal{X}} := |b(0)| + \left\| |b'(z)|^2\, dA \right\|_{CM(\mathcal{D})}^{1/2}$$

and denote by $\mathcal{X}_0$ the norm closure in $\mathcal{X}$ of the space of polynomials.

Our main result is

Theorem 1.

(1) *T_b is bounded if and only if $b \in \mathcal{X}$. In that case*

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\mathcal{X}}.$$

(2) *T_b is compact if and only if $b \in \mathcal{X}_0$.*

This result, which had been conjectured by the second author for some time, is part of an intriguing pattern of results involving boundedness of Hankel forms on Hardy spaces in one and several variables and boundedness of Schrödinger operators on the Sobolev space. We recall those results in the next subsection.

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Boundedness criteria for bilinear forms can be recast as weak factorization of function spaces and we discuss that in the third subsection. The first statement in Theorem 1 is equivalent to a weak factorization of the predual of $\mathcal{X}$; in notation we introduce there

$$(\mathcal{D} \odot \mathcal{D})^* = \mathcal{X}. \tag{1.1}$$

In the final subsection we describe the relation between Theorem 1 and classical results about Hankel matrices.

The proof of Theorem 1 is in Sections 2 and 3. It is easy to see that $\|T_b\|_{\mathcal{D}\times\mathcal{D}} \leq C\|b\|_{\mathcal{X}}$. To obtain the other inequality we must use the boundedness of T_b to show $|b'|^2\, dA$ is a Carleson measure. Analysis of the capacity theoretic characterization of Carleson measures due to Stegenga allows us to focus attention on a certain set V in $\mathbb{D}$ and the relative sizes of $\int_V |b'|^2$ and the capacity of the set $\bar{V} \cap \partial\bar{\mathbb{D}}$. To compare these quantities we construct V_{exp}, an expanded version of the set V which satisfies two conflicting conditions. First, V_{exp} is not much larger than V, either when measured by $\int_{V_{\mathrm{exp}}} |b'|^2$ or by the capacity of the $\overline{V_{\mathrm{exp}}} \cap \partial\bar{\mathbb{D}}$. Second, $\mathbb{D} \setminus V_{\mathrm{exp}}$ is well separated from V in a way that allows the interaction of quantities supported on the two sets to be controlled. Once this is done we can construct a function $\Phi_V \in \mathcal{D}$ which is approximately one on V and which has Φ_V' approximately supported on $\mathbb{D}\setminus V_{\mathrm{exp}}$. Using Φ_V we build functions f and g with the property that

$$|T_b(f,g)| = \int_V |b'|^2 + \text{ error.}$$

The technical estimates on Φ_V allow us to show that the error term is small and the boundedness of T_b then gives the required control of $\int_V |b'|^2$.

Once the first part of the theorem is established, the second follows rather directly.

1.2. **Other Bilinear Forms.** The Hardy space of the unit disk, $H^2(\mathbb{D})$, can be defined as the space of holomorphic functions on the disk with inner product

$$\langle f,g\rangle_{H^2(\mathbb{D})} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\,(1-|z|^2)dA$$

and normed by $\|f\|^2_{H^2(\mathbb{D})} = \langle f,f\rangle_{H^2(\mathbb{D})}$. Given a holomorphic *symbol function* b the Hankel form with symbol b is the bilinear form

$$T_b^{H^2(\mathbb{D})}(f,g) := \langle fg, b\rangle_{H^2(\mathbb{D})}. \tag{1.2}$$

The boundedness criteria for such forms was given by Nehari in 1957 [13]. He used the fact that functions in the Hardy space H^1 can be written as the product of functions in H^2 and showed $T_b^{H^2(\mathbb{D})}$ will be bounded if and only if b is in the dual space of H^1. Using Ch. Fefferman's identification of the dual of H^1 we can reformulate this in the language of Carleson measures. We say a positive measure μ on the disk is a *Carleson measure for* $H^2(\mathbb{D})$ if

$$\|\mu\|_{CM(H^2(\mathbb{D}))} := \sup\left\{\int_{\mathbb{D}} |f|^2\, d\mu : \|f\|_{H^2(\mathbb{D})} = 1\right\} < \infty.$$

The form $T_b^{H^2(\mathbb{D})}$ is bounded if and only if b is in the function space BMO or, equivalently, if and only if

$$|b'(z)|^2\,(1-|z|^2)dA \in CM(H^2(\mathbb{D})).$$

Later, in [8], Nehari's theorem was viewed as a result about Calderón-Zygmund singular integrals on spaces of homogenous type and an analogous result was proved for $H^2(\partial\mathbb{B}^n)$, the Hardy space of the sphere in complex $n-$space. In that context the Hankel form is defined similarly

$$T_b^{H^2(\partial\mathbb{B}^n)}(f,g) := \langle fg, b\rangle_{H^2(\partial\mathbb{B}^n)}.$$

That form is bounded if and only if b is in $BMO(\partial\mathbb{B}^n)$ or, equivalently, if and only if, with ∇ denoting the invariant gradient on the ball,

$$|\nabla b(z)|^2 \, dV \in CM\left(H^2(\partial\mathbb{B}^n)\right).$$

The approach in [8] is not well suited for analysis on the Hardy space of the polydisk, $H^2(\mathbb{D}^n)$. However Ferguson, Lacey, and Terwilleger were able to extend methods of multivariable harmonic analysis and obtain a result for $H^2(\mathbb{D}^n)$ [10], [11]. They showed that a Hankel form on $H^2(\mathbb{D}^n)$, again defined as a form whose value only depends on the product of its arguments, is bounded if and only if the symbol function b lies in $BMO(\mathbb{D}^n)$ or, equivalently, if and only if derivatives of b can be used to generate a Carleson measure for $H^2(\mathbb{D}^n)$.

In [12] Maz'ya and Ververbitsky presented a boundedness criterion for a bilinear form associated to the Schrödinger operator. Although their viewpoint and proof techniques were quite different from those used for Hankel forms, their result is formally very similar. We change their formulation slightly to make the analogy more visible, our b is related to their V by $b = -\Delta^{-1}V$. Let $\mathring{L}_2^1(\mathbb{R}^n)$ be the energy space (homogenous Sobolev space) obtained by completing $C_0^\infty(\mathbb{R}^n)$ with respect to the quasinorm induced by the Dirichlet inner product

$$\langle f,g\rangle_{\mathrm{Dir}} = \int_{\mathbb{R}^n} \nabla f \cdot \overline{\nabla g} \, dx.$$

Given b, a bilinear Schrödinger form on $\mathring{L}_2^1(\mathbb{R}^n) \times \mathring{L}_2^1(\mathbb{R}^n)$ is defined by

$$S_b(f,g) = \langle fg, b\rangle_{\mathrm{Dir}}.$$

We will say a measure μ on $\mathbb{R}^n$ is a *Carleson measure for the energy space* if

$$\|\mu\|_{CM(\mathring{L}_2^1(\mathbb{R}^n))} := \sup\left\{\int_{\mathbb{R}^n} |f|^2 \, d\mu : \|f\|_{\mathring{L}_2^1(\mathbb{R}^n)} = 1\right\} < \infty.$$

Corollary 2 of [12] is that S_b is bounded if and only if

$$|(-\Delta)^{1/2} b|^2 dx \in CM(\mathring{L}_2^1(\mathbb{R}^n)).$$

It would be very satisfying to know an underlying reason for the similarity of these various results to each other and to Theorem 1.

1.3. **Reformulation in Terms of Weak Factorization.** In his proof Nehari used the fact that any function $f \in H^1(\mathbb{D})$ could be factored as $f = gh$ with $g, h \in H^2(\mathbb{D})$, $\|f\|_{H^1(\mathbb{D})} = \|g\|_{H^2(\mathbb{D})} \|h\|_{H^2(\mathbb{D})}$. In [8] the authors develop a weak substitute for this. For two Banach spaces of functions, $\mathcal{A}$ and $\mathcal{B}$, defined on the same domain, define the weakly factored space $\mathcal{A} \odot \mathcal{B}$ to be the completion of finite sums $f = \sum a_i b_i$; $\{a_i\} \subset \mathcal{A}$, $\{b_i\} \subset \mathcal{B}$ using the norm

$$\|f\|_{\mathcal{A}\odot\mathcal{B}} = \inf\left\{\sum \|a_i\|_{\mathcal{A}} \|b_i\|_{\mathcal{B}} : f = \sum a_i b_i\right\}.$$

It is shown in [8] that $H^2(\partial\mathbb{B}^n)\odot H^2(\partial\mathbb{B}^n) = H^1(\partial\mathbb{B}^n)$ and consequentially

$$\left(H^2(\partial\mathbb{B}^n)\odot H^2(\partial\mathbb{B}^n)\right)^* = BMO(\partial\mathbb{B}^n). \tag{1.3}$$

(In this context, by "=" we mean equality of the function spaces and equivalence of the norms.) Based on the analogy between (1.1) and (1.3) we think of $\mathcal{D}\odot\mathcal{D}$ as a type of H^1 space and of $\mathcal{X}$ as a type of BMO space. That viewpoint is developed further in [4].

The precise formulation of (1.1) is the following corollary.

Corollary 1. *For $b\in\mathcal{X}$ set $\Lambda_b h = T_b(h,1)$, then $\Lambda_b\in(\mathcal{D}\odot\mathcal{D})^*$. Conversely, if $\Lambda\in(\mathcal{D}\odot\mathcal{D})^*$ there is a unique $b\in\mathcal{X}$ so that for all $h\in\mathcal{P}(\mathbb{D})$ we have $\Lambda h = T_b(h,1) = \Lambda_b h$. In both cases $\|\Lambda_b\|_{(\mathcal{D}\odot\mathcal{D})^*}\approx\|b\|_{\mathcal{X}}$.*

Proof. If $b\in\mathcal{X}$ and $f\in\mathcal{D}\odot\mathcal{D}$, say $f=\sum g_ih_i$ with $\sum\|g_i\|_{\mathcal{D}}\|h_i\|_{\mathcal{D}}\leq\|f\|_{\mathcal{D}\odot\mathcal{D}}+\varepsilon$, then

$$\begin{aligned}|\Lambda_b f| &= \left|\sum_{i=1}^\infty\langle g_ih_i,b\rangle_{\mathcal{D}}\right| = \left|\sum_{i=1}^\infty T_b(g_i,h_i)\right|\\ &\leq\|T_b\|\sum_{i=1}^\infty\|g_i\|_{\mathcal{D}}\|h_i\|_{\mathcal{D}}\leq\|T_b\|\left(\|f\|_{\mathcal{D}\odot\mathcal{D}}+\varepsilon\right).\end{aligned}$$

It follows that $\Lambda_b f = \langle f,b\rangle_{\mathcal{D}}$ defines a continuous linear functional on $\mathcal{D}\odot\mathcal{D}$ with $\|\Lambda_b\|\leq\|T_b\|$.

Conversely, if $\Lambda\in(\mathcal{D}\odot\mathcal{D})^*$with norm $\|\Lambda\|$, then for all $f\in\mathcal{D}$

$$|\Lambda f| = |\Lambda(f\cdot 1)|\leq\|\Lambda\|\,\|f\|_{\mathcal{D}}\|1\|_{\mathcal{D}} = \|\Lambda\|\,\|f\|_{\mathcal{D}}.$$

Hence there is a unique $b\in\mathcal{D}$ such that $\Lambda f=\Lambda_b f$ for $f\in\mathcal{D}$. Finally, if $f=gh$ with $g,h\in\mathcal{D}$ we have

$$\begin{aligned}|T_b(g,h)| &= |\langle gh,b\rangle_{\mathcal{D}}| = |\Lambda_b f| = |\Lambda f|\\ &\leq\|\Lambda\|\,\|f\|_{\mathcal{D}\odot\mathcal{D}}\leq\|\Lambda\|\,\|g\|_{\mathcal{D}}\|h\|_{\mathcal{D}},\end{aligned}$$

which shows that T_b extends to a continuous bilinear form on $\mathcal{D}\odot\mathcal{D}$ with $\|T_b\|\leq\|\Lambda\|$. By Theorem 1 we conclude $b\in\mathcal{X}$ and collecting the estimates that $\|\Lambda\| = \|\Lambda_b\|_{(\mathcal{D}\odot\mathcal{D})^*}\approx\|T_b\|\approx\|b\|_{\mathcal{X}}$. □

Define the space $\partial^{-1}(\partial\mathcal{D}\odot\mathcal{D})$ to be the completion of the space of functions f which have $f'=\sum_{i=1}^N g_i'h_i$ (and thus $f=\partial^{-1}\sum(\partial g_i)h_i)$) using the norm

$$\|f\|_{\partial^{-1}(\partial\mathcal{D}\odot\mathcal{D})} = \inf\left\{\sum\|g_i\|_{\mathcal{D}}\|h_i\|_{\mathcal{D}} : f'=\sum_{i=1}^N g_i'h_i\right\}.$$

Using the previous corollary we can recapture, but by a very indirect route, an earlier result of Coifman-Murai [9], Tolokonnikov [17], and Rochberg-Wu [15].

Corollary 2 ([9], [17], [15])**.**

$$\left(\partial^{-1}(\partial\mathcal{D}\odot\mathcal{D})\right)^* = \mathcal{X}.$$

Proof. As in the previous proof this statement is equivalent to a boundedness criterion for a class of bilinear forms. In this case the forms of interest are those defined on $\mathcal{D}\times\mathcal{D}$ by

$$K_b(f,g) = \int_{\mathbb{D}} f'g\overline{b'}dV.$$

The proof given later that T_b is bounded if $b \in \mathcal{X}$ in fact shows that K_b is bounded and then notes that

$$T_b(f,g) = K_b(f,g) + K_b(g,f) + (fg\bar{b})(0). \tag{1.4}$$

In the other direction, if K_b is bounded then the same relation shows T_b is bounded and we can then appeal to Theorem 1. □

The proofs in [9], [17], and [15] give, explicitly or implicitly, estimates from below for $|K_b(f,g)|$. In proving Theorem 1 we need to estimate $|T_b(f,g)|$ from below. We avoided using the representation (1.4) as a starting point because it was unclear how to analyze the potential cancellation between terms on the right hand side of (1.4).

Combining the previous corollaries we have, with the obvious notation,

Corollary 3.

$$\partial\left(\mathcal{D} \odot \mathcal{D}\right) = \partial\mathcal{D} \odot \mathcal{D}.$$

In contrast

$$\partial\left(\mathcal{D} \odot \mathcal{D}\right) \neq \partial^{1/2}\mathcal{D} \odot \partial^{1/2}\mathcal{D}.$$

To see this note that $\partial^{1/2}\mathcal{D} \odot \partial^{1/2}\mathcal{D} = H^2\left(\mathbb{D}\right) \odot H^2\left(\mathbb{D}\right) = H^1\left(\mathbb{D}\right)$ and that $f(z) = \left(\log\left(1-z\right)\right)^{3/2}$ satisfies $f' \in \partial\left(\mathcal{D} \odot \mathcal{D}\right)$, $f' \notin H^1$.

1.4. **Reformulation in Terms of Matrices.** If T_b is given by (1.2) with $b(z) = \sum b_n z^n$ then the matrix representation of T_b with respect to the monomial basis is $\left(\bar{b}_{i+j}\right)$. Nehari's theorem gives a boundedness condition for such Hankel matrices; matrices $(a_{i,j})$ for which $a_{i,j}$ is a function of $i+j$. There are analogous results for Hankel forms on Bergman spaces. Those forms have matrices

$$\Big((i+1)^{\alpha}\,(j+1)^{\beta}\,(i+j+1)^{\gamma}\,\bar{b}(i+j)\Big) \tag{1.5}$$

with $\alpha, \beta > 0$ and are bounded if and only if $b(z)$ is in the Bloch space. The criteria for (1.5) to belong to the Schatten–von Neumann classes is known if $\min\{\alpha,\beta\} > -1/2$ and it is known that those results do not extend to $\min\{\alpha,\beta\} \leq -1/2$. For all of this see [14, Ch 6.8].

The matrix representations of the forms T_b and K_b with respect the the basis of normalized monomials of $\mathcal{D}$ are of the form (1.5) with (α,β) equal to $(-1/2,-1/2)$ in the first case and $(-1/2,1/2)$ in the second.

2. Preliminary Steps in the Proof

2.1. **The Proof of (2) Given (1).** Suppose T_b is compact. For any holomorphic function $k(z)$ on $\mathbb{D}$ and r, $0 < r < 1$, set $S_r k(z) = k(rz)$. A computation with monomials verifies that

$$T_{S_r b}\left(f,g\right) = T_b(S_r f, S_r g).$$

As $r \to 1$, S_r converges strongly to I. Using this and the fact that T_b is compact we obtain $\lim \|T_{S_r b} - T_b\| = 0$. Hence, by the first part of the theorem $\lim \|S_r b - b\|_{\mathcal{X}} = 0$. The Taylor coefficients of $S_r b$ decay geometrically, hence $S_r b \in \mathcal{X}_0$ and thus $b \in \mathcal{X}_0$.

In the other direction note that if b is a polynomial then T_b is finite rank and hence compact. If $\{b_n\} \subset \mathcal{P}(\mathbb{D})$ is a sequence of polynomials which converge in norm to $b \in \mathcal{X}_0$ then, by the first part of the theorem T_b is the norm limit of the T_{b_n} and hence is also compact.

2.2. **The Proof of The Easy Direction of (1).** Suppose that μ_b is a $\mathcal{D}$-Carleson measure. For $f,g\in\mathcal{P}(\mathbb{D})$ we have

$$\begin{aligned}
|T_b(f,g)| &= \left| f(0)g(0)\overline{b(0)} + \int_{\mathbb{D}} (f'(z)g(z)+f(z)g'(z))\overline{b'(z)}dA \right| \\
&\le |f(0)g(0)b(0)| + \int_{\mathbb{D}} |f'(z)g(z)b'(z)|\,dA + \int_{\mathbb{D}} |f(z)g'(z)b'(z)|\,dA \\
&\le |(fgb)(0)| + \|f\|_{\mathcal{D}}\left(\int_{\mathbb{D}}|g|^2\,d\mu_b\right)^{1/2} + \|g\|_{\mathcal{D}}\left(\int_{\mathbb{D}}|f|^2\,d\mu_b\right)^{1/2} \\
&\le C\left(|b(0)| + \|\mu_b\|_{\mathcal{D}-Carleson}\right)\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}} \\
&= C\|b\|_{\mathcal{X}}\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}.
\end{aligned}$$

Thus T_b has a bounded extension to $\mathcal{D}\times\mathcal{D}$ with $\|T_b\|\le C\|b\|_{\mathcal{X}}$.

We note for later that if T_b extends to a bounded bilinear form on $\mathcal{D}$ then $b\in\mathcal{D}$, equivalently, $d\mu_b$ is a finite measure. To see this note that for all $f\in\mathcal{P}(\mathbb{D})$, $|\langle f,b\rangle_{\mathcal{D}}| = |T_b(f,1)| \le \|T_b\|\|f\|_{\mathcal{D}}\|1\|_{\mathcal{D}}$. Thus $b\in\mathcal{D}$ and

$$\|b\|_{\mathcal{D}} \le C\|T_b\|. \tag{2.1}$$

2.3. **Disk Capacity and Disk Blowups.** To complete the proof of Theorem 1 we must show that if T_b is bounded then $\mu_{b:} = |b'|^2\,dA$ is a $\mathcal{D}$-Carleson measure. We will do this by showing that μ_b satisfies a capacitary condition introduced by Stegenga [16].

For an interval I in the circle we let I_m be its midpoint and $z(I) = (1-|I|/2\pi)\,I_m$ be the associated index point in the disk. In the other direction let $I(z)$ be the interval such that $z(I(z)) = z$. Let $T(I)$ be the tent over I, the convex hull of I and $z(I)$ and let $T(z) = T(z(I)) := T(I)$. More generally, for any open subset H of the circle $\mathbb{T}$, we define $T(H)$, the *tent region* of H in the disk $\mathbb{D}$, by

$$T(H) = \bigcup_{I\subset H} T(I).$$

For G in the circle $\mathbb{T}$ define the capacity of G by

$$Cap_{\mathbb{D}}G = \inf\left\{\|\psi\|_{\mathcal{D}}^2 : \psi(0)=0, \operatorname{Re}\psi(z)\ge 1 \text{ for } z\in G\right\}. \tag{2.2}$$

Stegenga [16] has shown that μ is a $\mathcal{D}$-Carleson measure exactly if for any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu\left(\bigcup_{j=1}^N T(I_j)\right) \le C\;Cap_{\mathbb{D}}\left(\bigcup_{j=1}^N I_j\right). \tag{2.3}$$

We will need to understand how the capacity of a set changes if we expand it in certain ways. For I an open arc and $0<\rho\le 1$, let I^ρ be the arc concentric with I having length $|I|^\rho$.

Definition 1 (disk blowup). *For G open in $\mathbb{T}$ we call*

$$G_{\mathbb{D}}^\rho = \bigcup_{I\subset G} T(I^\rho)$$

the disk blowup *(of order ρ) of G.*

The important feature of the disk blowup is that it achieves a good geometric separation between $\mathbb{D}\setminus G_{\mathbb{D}}^{\rho}$ and $G_{\mathbb{D}}^{1}=T\left(G\right)$. This plays a crucial role in using Schur's test to estimate an integral later, as well as in estimating an error term near the end of the paper.

Lemma 1. *Let G be an open subset of the circle $\mathbb{T}$. If $w\in G_{\mathbb{D}}^{1}=T\left(G\right)$ and $z\notin G_{\mathbb{D}}^{\rho}$ then $|z-w|\geq(1-|w|^{2})^{\rho}$.*

Proof. The inequality follows from the definition of $G_{\mathbb{D}}^{\rho}$ and the inclusion $T\left(I^{\rho}\right)\subset\left\{z:|z-z\left(I\right)|<2(1-|z\left(I\right)|)^{2\rho}\right\}$. □

It would be useful to us if we knew there were constants C_{ρ}, $0<\rho<1$, such that

$$Cap_{\mathbb{D}}\left(\bigcup_{I\subset G}I^{\rho}\right)\leq C_{\rho}Cap_{\mathbb{D}}G. \tag{2.4}$$

and

$$\lim_{\rho\rightarrow1^{-}}C_{\rho}=1. \tag{2.5}$$

Bishop proved (2.4) [6] but did not obtain (2.5) and we could not obtain it directly. In the next subsection we obtain Lemma 4, an analog of (2.4) and (2.5) in a tree model, and that will play an important role in the proof. After we show that tree and disk are comparible, Corollary 5, then we will also have (2.5).

2.4. **Tree Capacity and Tree Blowups.** In our study of capacities and approximate extremals it will sometimes be convenient to transfer our arguments to and from the Bergman tree $\mathcal{T}$ and to work with the associated tree capacities. We now recall the notation associated to $\mathcal{T}$. Further properties of $\mathcal{T}$ are in the Appendix and a more extensive investigation with other applications is in [5].

Let $\mathcal{T}$ be the standard Bergman tree in the unit disk $\mathbb{D}$. That is $\mathcal{T}=\{x\}$ is the index set for the subsets $\{B_x\}$ of $\mathbb{D}$ obtained by decomposing $\mathbb{D}$, first with the circles $C_k=\left\{z:|z|=1-2^{-k}\right\}$, $k=1,2,...$ and then for each k making 2^k radial cuts in the ring bounded by C_k and C_{k+1}. We refer to the $\{B_x\}$ as boxes and we emphasize the standard bijection between the boxes and the intervals on the circle $\{I(B_x)\}$ obtained by radial projection of the boxes. This also induces a bijection with the point set $\{z(I(B_x))\}$ in the disk, furthermore $z(I(B_x))\in B_x$. At times we will use the label x to denote the point $z(I(B_x))$.

$\mathcal{T}$ is a rooted dyadic tree with root $\{0\}$ which we denote o. For a vertex x of $\mathcal{T}$ we denote its immediate predecessor by x^{-1} and its two immediate successors by x_+ and x_-. We let $d(x)$ equal the number of nodes on the geodesic $[o,x]$. The successor set of x is $S(x)=\{y\in\mathcal{T}:y\geq x\}$.

We say that $S\subset\mathcal{T}$ is a *stopping time* if no pair of distinct points in S are comparable in $\mathcal{T}$. Given stopping times $E,F\subset\mathcal{T}$ we say that $F\succ E$ if for every $x\in F$ there is $y\in E$ above x, i.e., with $x>y$. For stopping times $F\succ E$ denote by $\mathcal{G}\left(E,F\right)$ the union of all those geodesics connecting a point of $x\in F$ to the point $y\in E$ above it.

The bijections between $\{B_x\}$, $\{I(B_x)\}$, and $\{z(I(B_x))\}$ induce bijections between other sets. We will be particularly interested in three types of sets:

- *stopping times* W in the tree $\mathcal{T}$;
- $\mathcal{T}$-*open subsets* G of the circle $\mathbb{T}$;

- $\mathcal{T}$-*tent regions* Γ of the disk $\mathbb{D}$.

The bijections are given as follows. For W a *stopping time* in $\mathcal{T}$, its associated $\mathcal{T}$-*open set* in $\mathbb{T}$ is the $\mathcal{T}$-shadow $S_{\mathcal{T}}(W) = \cup\{I(x) : x \in W\}$ of W on the circle (this also *defines* the collection of $\mathcal{T}$-open sets). The associated $\mathcal{T}$-*tent region* in $\mathbb{D}$ is $T_{\mathcal{T}}(W) = \cup\{T(I(\kappa)) : \kappa \in W\}$ (this also *defines* the collection of $\mathcal{T}$-tent regions).

At times we will identify a stopping time $W = W_{\mathcal{T}}$ in a tree $\mathcal{T}$ with its associated $\mathcal{T}$-shadow on the circle and its $\mathcal{T}$-tent region in the disk and will use W or $W_{\mathcal{T}}$ to denote any of them. When we do this the exact interpretation will be clear from the context.

Note that for any open subset E of the circle $\mathbb{T}$, there is a unique $\mathcal{T}$-open set $G \subset E$ such that $E \setminus G$ is at most countable. We often informally identify the open sets E and G.

For a functions k, K defined on $\mathcal{T}$ set

$$Ik(x) = \sum_{y \in [o,x]} k(y), \ \Delta K(x) = K(x) - K(x^-)$$

with the convention that $K(o^-) = 0$.

For $\Omega \subseteq \mathcal{T}$ a point $x \in \mathcal{T}$ is in the interior of Ω if $x, x^{-1}, x_+, x_- \in \Omega$. A function H is *harmonic* in Ω if

$$H(x) = \frac{1}{3}[H(x^{-1}) + H(x_+) + H(x_-)] \tag{2.6}$$

for every point x which is interior in Ω. If $H = Ih$ is harmonic then for all x in the interior of Ω

$$h(x) = h(x_+) + h(x_-). \tag{2.7}$$

Let $Cap_{\mathcal{T}}$ be the tree capacity associated with $\mathcal{T}$:

$$Cap_{\mathcal{T}}(E) = \inf\left\{\|f\|^2_{\ell^2(\mathcal{T})} : If \geq 1 \text{ on } E\right\}. \tag{2.8}$$

More generally, if $E, F \subset \mathcal{T}$ are disjoint stopping times with $F \succ E$, the capacity of the pair (E, F), commonly known as a condenser, is given by

$$Cap_{\mathcal{T}}(E, F) = \inf\left\{\|f\|^2_{\ell^2(\mathcal{T})} : If \geq 1 \text{ on } F, \ \text{supp}(f) \subset \bigcup_{e \in E} S(e)\right\}. \tag{2.9}$$

Let $\mathcal{T}_\theta$ be the rotation of the tree $\mathcal{T}$ by the angle θ, and let $Cap_{\mathcal{T}_\theta}$ be the tree capacity associated with $\mathcal{T}_\theta$ as in (2.8), and extend the definition to open subsets G of the circle $\mathbb{T}$ by,

$$Cap_{\mathcal{T}_\theta}(G) = \inf\left\{\sum_{\kappa \in \mathcal{T}_\theta} f(\kappa)^2 : If(\beta) \geq 1 \text{ for } \beta \in \mathcal{T}_\theta, \, I(\beta) \subset G\right\}.$$

This is consistent with the definition of tree capacity of a stopping time W in $\mathcal{T}_\theta$; that is, if $G = \cup\{I(\kappa) : \kappa \in W\}$ we have

$$Cap_{\mathcal{T}_\theta}(W) = Cap_{\mathcal{T}_\theta}(\{o\}, W) = Cap_{\mathcal{T}_\theta}(G).$$

When the angle θ is not important, we will simply write $\mathcal{T}$ with the understanding that all results have analogues with $\mathcal{T}_\theta$ in place of $\mathcal{T}$.

We will use functions on the disk which are approximate extremals for measuring capacity, that is functions for which the equality in (2.2) is approximately attained. A tool in doing that is an analysis of the model problems on a tree. The following result about tree capacities and extremals is proved in the Appendix.

Proposition 1. *Suppose E, $F \subset \mathcal{T}$ are disjoint stopping times with $F \succ E$.*

1. *There is an extremal function $H = Ih$ such that $Cap(E, F) = \|h\|_{\ell^2}^2$.*
2. *The function H is harmonic on $\mathcal{T} \setminus (E \cup F)$.*
3. *If S is a stopping time in $\mathcal{T}$, then $\sum_{\kappa \in S} |h(\kappa)| \leq 2Cap(E, F)$.*
4. *The function h is positive on $\mathcal{G}(E, F)$, and zero elsewhere.*

Definition 2 (stopping time blowup)**.** *Given $0 \leq \rho \leq 1$ and a stopping time W in a tree $\mathcal{T}$, define the* stopping time blowup $W_{\mathcal{T}}^{\rho}$ *of W in $\mathcal{T}$ as the set of minimal tree elements in $\{R^{\rho}\kappa : \kappa \in \mathcal{T}_{\theta}\}$, where $R^{\rho}\kappa$ denotes the unique element in the tree $\mathcal{T}$ satisfying*

$$\begin{gathered} o \leq R^{\rho}\kappa \leq \kappa, \\ \rho d(\kappa) \leq d(R^{\rho}\kappa) < \rho d(\kappa) + 1. \end{gathered} \tag{2.10}$$

Clearly $W_{\mathcal{T}}^{\rho}$ is a stopping time in $\mathcal{T}$. Note that $R^1\kappa = \kappa$. The element $R^{\rho}\kappa$ can be thought of as the "ρ^{th} root of κ" since $|R^{\rho}\kappa| = 2^{-d(R^{\rho}\kappa)} \approx 2^{-\rho d(\kappa)} = |\kappa|^{\rho}$.

If W is a stopping time for $\mathcal{T}$ and $W_{\mathcal{T}}^{\rho}$ is the stopping time blowup of W, then there is a good estimate for the tree capacity of $W_{\mathcal{T}}^{\rho}$ given in Lemma 4 below: $Cap_{\mathcal{T}}(\{o\}, W_{\mathcal{T}}^{\rho}) \leq \rho^{-2} Cap_{\mathcal{T}}(\{o\}, W)$. Unfortunately there is not a good condenser estimate of the form $Cap_{\mathcal{T}}(W_{\mathcal{T}}^{\rho}, W) \leq C_{\rho} Cap_{\mathcal{T}}(\{o\}, W)$; the left side can be infinite when the right side is finite. We now introduce another type of blowup, a tree analog of the disk blowup, for which we do have an effective condenser estimate. We do this using a capacitary extremal function and a comparison principle. Let W be a stopping time in $\mathcal{T}$. By Proposition 1, there is a unique extremal function $H = Ih$ such that

$$\begin{gathered} Ih(x) = H(x) = 1 \text{ for } x \in W, \\ Cap_{\mathcal{T}} W = \|h\|_{\ell^2}^2 . \end{gathered} \tag{2.11}$$

Definition 3 (capacitary blowup)**.** *Given a stopping time W in $\mathcal{T}$, the corresponding extremal H satisfying (2.11), and $0 < \rho < 1$, define the* capacitary blowup $\widehat{W_{\mathcal{T}}^{\rho}}$ *of W by*

$$\widehat{W_{\mathcal{T}}^{\rho}} = \{t \in \mathcal{G}(\{o\}, W) : H(t) \geq \rho \text{ and } H(x) \leq \rho \text{ for } x < t\}.$$

Clearly $\widehat{W_{\mathcal{T}}^{\rho}}$ is a stopping time in $\mathcal{T}$.

Lemma 2. $Cap_{\mathcal{T}} \widehat{W_{\mathcal{T}}^{\rho}} \leq \rho^{-2} Cap_{\mathcal{T}} W$.

Proof. Let H be the extremal for W in (2.11) and set $h = \Delta H$, $h^{\rho} = \frac{1}{\rho} h$ and $H^{\rho} = \frac{1}{\rho} H$. Then H^{ρ} is a candidate for the infimum in the definition of capacity of $\widehat{W_{\mathcal{T}}^{\rho}}$, and hence by the "comparison principle",

$$Cap_{\mathcal{T}} \widehat{W_{\mathcal{T}}^{\rho}} \leq \|h^{\rho}\|_{\ell^2}^2 = \left(\frac{1}{\rho}\right)^2 \|h\|_{\ell^2}^2 = \rho^{-2} Cap_{\mathcal{T}} W.$$

□

The next lemma is used in the proof of our main estimate, (3.1) and it requires an upper bound on $Cap_{\mathbb{D}}(G)$. However (3.1) is straightforward if $Cap_{\mathbb{D}}(G)$ bounded away from zero so that restriction is not a problem. In fact, moving forward we will assume, at times implicitly, that $Cap_{\mathbb{D}}(G)$ is not large.

Lemma 3. *$Cap_{\mathcal{T}}\left(W,\widehat{W_{\mathcal{T}}^{\rho}}\right)\leq\frac{4}{(1-\rho)^2}Cap_{\mathcal{T}}W$ provided $Cap_{\mathcal{T}}W\leq(1-\rho)^2/4$.*

Proof. Let H be the extremal for W in (2.11). For $t\in\widehat{W_{\mathcal{T}}^{\rho}}$ we have by our assumption,

$$h(t)\leq\|h\|_{\ell^2}\leq\sqrt{Cap_{\mathcal{T}}W}\leq\frac{1}{2}(1-\rho),$$

and so

$$H(t)=H\left(t^-\right)+h(t)\leq\rho+\frac{1}{2}(1-\rho)=\frac{1+\rho}{2}.$$

If we define $\widetilde{H}(t)=\frac{2}{1-\rho}\left\{H(t)-\frac{1+\rho}{2}\right\}$, then $\widetilde{H}\leq 0$ on $\widehat{W_{\mathcal{T}}^{\rho}}$ and $\widetilde{H}=1$ on W. Thus $\widetilde{H}$ is a candidate for the capacity of the condenser and so by the "comparison principle"

$$\begin{aligned}Cap_{\mathcal{T}}\left(W,\widehat{W_{\mathcal{T}}^{\rho}}\right)&\leq\left\|\triangle\widetilde{H}\right\|^2_{\ell^2\left(\mathcal{G}\left(W_{\mathcal{T}}^{\rho},W\right)\right)}\leq\left\|\triangle\widetilde{H}\right\|^2_{\ell^2(\mathcal{T})}\\&=\left(\frac{2}{1-\rho}\right)^2\|h\|^2_{\ell^2(\mathcal{T})}=\frac{4}{(1-\rho)^2}Cap_{\mathcal{T}}W.\end{aligned}$$

□

We also have good *tree* separation inherited from the stopping time blowup $W_{\mathcal{T}}^{\rho}$. This gives our substitute for (2.4) and (2.5).

Lemma 4. *$W_{\mathcal{T}}^{\rho}\subset\widehat{W_{\mathcal{T}}^{\rho}}$ as open subsets of the circle or, equivalently, as $\mathcal{T}$-tent regions in the disk. Consequently $Cap_{\mathcal{T}}W_{\mathcal{T}}^{\rho}\leq\rho^{-2}Cap_{\mathcal{T}}W$.*

Proof. The restriction of H to a geodesic is a concave function of distance from the root, and so if $o<z<w\in W$, then

$$H(z)\geq\left(1-\frac{d(z)}{d(w)}\right)H(o)+\frac{d(z)}{d(w)}H(w)=\frac{d(z)}{d(w)}\geq\rho,\ z\in\widehat{W_{\mathcal{T}}^{\rho}},$$

and this proves $W_{\mathcal{T}}^{\rho}\subset\widehat{W_{\mathcal{T}}^{\rho}}$. The inequality now follows from Lemma 2. □

2.5. **Holomorphic Approximate Extremals and Capacity Estimates.** We now define a holomorphic approximation Φ to the extremal function $H=Ih$ on $\mathcal{T}$ constructed in Proposition 1. We will use a parameter s. We always suppose $s>-1$ and additional specific assumptions will be made at various places. Define $\varphi_\kappa(z)=\left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s}$ and set

$$\Phi(z)=\sum_{\kappa\in\mathcal{T}}h(\kappa)\varphi_\kappa(z)=\sum_{\kappa\in\mathcal{T}}h(\kappa)\left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s}.\tag{2.12}$$

Note that for $\tau\in\mathcal{T}$

$$\sum_{\kappa\in\mathcal{T}}h(\kappa)I\delta_\kappa(\tau)=I\left(\sum_{\kappa\in\mathcal{T}}h(\kappa)\delta_\kappa\right)(\tau)=Ih(\tau)=H(\tau),$$

and so

$$\Phi(z)-H(z)=\sum_{\kappa\in\mathcal{T}}h(\kappa)\left\{\varphi_\kappa-I\delta_\kappa\right\}(z).\tag{2.13}$$

Define Γ_s by

$$\Gamma_s h(z) = \int_{\mathbb{D}} h(\zeta) \frac{(1-|\zeta|^2)^s}{\left(1-\overline{\zeta}z\right)^{1+s}} dA, \tag{2.14}$$

and recall that for appropriate constant c_s, $c_s\Gamma_s$ is a projection onto holomorphic functions [18, Thm 2.11]. For notational convenience we absorb the constant c_s into the measure dA. Thus for $h \in \mathcal{P}(\mathbb{D})$,

$$\Gamma_s h(z) = h(z). \tag{2.15}$$

We then have $\Phi = \Gamma_s g$ where

$$g(\zeta) = \sum_{\kappa\in\mathcal{T}} h(\kappa) \frac{1}{|B_\kappa|} \frac{\left(1-\overline{\zeta}\kappa\right)^{1+s}}{(1-|\zeta|^2)^s} \chi_{B_\kappa}(\zeta), \tag{2.16}$$

and B_κ is the Euclidean ball centered at κ with radius $c(1-|\kappa|)$ where c is a small positive constant to be chosen later. The function Φ satisfies the following estimates.

Proposition 2. *Set $F = \widehat{E^\rho_{\mathcal{T}}}$ and write $E = \{w_k\}_k$. Suppose $z \in \mathbb{D}$ and $s > -1$. Then we have*

$$\left\{ \begin{array}{lll} |\Phi(z) - \Phi(w_k)| & \leq & CCap_{\mathcal{T}}(E,F), \quad z \in T(w_k) \\ \operatorname{Re}\Phi(w_k) & \geq & c > 0, \quad k \geq 1 \\ |\Phi(w_k)| & \leq & C, \quad k \geq 1 \\ |\Phi(z)| & \leq & CCap_{\mathcal{T}}(E,F), \quad z \notin F. \end{array} \right. . \tag{2.17}$$

Corollary 4. *Furthermore, if $s > -\frac{1}{2}$ then $\Phi = \Gamma_s g$ for a g which satisfies*

$$\int_{\mathbb{D}} |g(\zeta)|^2 dA \leq C\ Cap_{\mathcal{T}}(E,F); \tag{2.18}$$

and if $s > \frac{1}{2}$ then

$$\|\Phi\|^2_{\mathcal{D}} \leq \int_{\mathbb{D}} |g(\zeta)|^2 dA \leq C\ Cap_{\mathcal{T}}(E,F). \tag{2.19}$$

Proof. From (2.13) we have

$$\begin{aligned} |\Phi(z) - H(z)| &\leq \sum_{\kappa\in[o,z]} |h(\kappa)\{\varphi_\kappa(z) - 1\}| + \sum_{\kappa\notin[o,z]} |h(\kappa)\varphi_\kappa(z)| \\ &= I(z) + II(z). \end{aligned}$$

We also have that h is nonnegative and supported in $V^\gamma_G \setminus V^\alpha_G$. We first show that

$$II(z) \leq \sum_{\kappa\notin[o,z]} h(\kappa) \left| \frac{1-|\kappa|^2}{1-\overline{\kappa}z} \right|^{1+s} \leq CCap(E,F).$$

For $A > 1$ let

$$\Omega_j = \left\{ \kappa \in \mathcal{T} : A^{-j-1} < \left| \frac{1-|\kappa|^2}{1-\overline{\kappa}z} \right| \leq A^{-j} \right\}.$$

Lemma 5. *For every j the set Ω_j is a union of two stopping times for $\mathcal{T}$.*

Proof. Let Ω_j^1 be the subset of Ω_j of points whose distance from the root is odd and set $\Omega_j^2 = \Omega_j \setminus \Omega_j^1$. We will show both are stopping times; i.e. if for $r = 1, 2$, $\kappa \in \Omega_j^r$, $\lambda \in \mathcal{T}$, and $\kappa \in [o, \lambda)$, then $\lambda \notin \Omega_j^r$.

Set $\delta\kappa = \lambda - \kappa$. We have

$$\begin{aligned} \left|\frac{1-\bar{\lambda}z}{1-|\lambda|^2}\right| &= \frac{1-|\kappa|^2}{1-|\lambda|^2}\left|\frac{1-\left(\overline{\kappa+\delta\kappa}\right)z}{1-|\kappa|^2}\right| \\ &= \frac{1-|\kappa|^2}{1-|\lambda|^2}\left|\frac{1-\bar{\kappa}z}{1-|\kappa|^2} - \frac{\overline{\delta\kappa}z}{1-|\kappa|^2}\right| \\ &\geq \frac{1-|\kappa|^2}{1-|\lambda|^2}\left\{\left|\frac{1-\bar{\kappa}z}{1-|\kappa|^2}\right| - \frac{\left|\overline{\delta\kappa}z\right|}{1-|\kappa|^2}\right\} \end{aligned} \tag{2.20}$$

By the construction of the tree $(1-|\kappa|^2) \sim 2^s(1-|\lambda|^2)$ for some positive integer s, and if κ and λ are in the same Ω_j^r then $s \geq 2$. Also, by the construction of $\mathcal{T}$, we have

$$\frac{\left|\overline{\delta\kappa}z\right|}{1-|\kappa|^2} \leq \frac{\sqrt{2}\,(1-|\kappa|)\,|z|}{1-|\kappa|^2} \lesssim \frac{\sqrt{2}}{2},$$

and hence we continue with

$$\left|\frac{1-\bar{\lambda}z}{1-|\lambda|^2}\right| \geq 4\left(A^j - \frac{\sqrt{2}}{2}\right).$$

We are done if for each j, $A^{j+1} \leq 4\left(A^j - \sqrt{2}/2\right)$. That holds if $A \leq 4(1-\sqrt{2}/2) < 1.17$. □

Now by the stopping time property, item 3 in Proposition 1, we have

$$\sum_{\kappa\in\Omega_j} h(\kappa) \leq C Cap_{\mathcal{T}}(E, F), \ j \geq 0.$$

Altogether we then have

$$II(z) \leq \sum_{j=0}^{\infty}\sum_{\kappa\in\Omega_j} h(\kappa)\, A^{-j(1+s)} \leq C_s Cap_{\mathcal{T}}(E, F).$$

If $z \in \mathbb{D}\setminus F$ then $I(z) = 0$ and $H(z) = 0$ and we have

$$|\Phi(z)| = |\Phi(z) - H(z)| \leq II(z) \leq C_s Cap_{\mathcal{T}}(E, F),$$

which is the fourth line in (2.17).

If $z \in T(w_j)$, then for $\kappa \notin [o, w_j]$ we have

$$|\varphi_\kappa(w_j)| \leq C\,|\varphi_\kappa(z)|,$$

and for $\kappa \in [o, z]$ we have

$$|\varphi_\kappa(z) - \varphi_\kappa(w_j)| = \left|\left(\frac{1-|\kappa|^2}{1-\overline{\kappa}z}\right)^{1+s} - \left(\frac{1-|\kappa|^2}{1-\overline{\kappa}w_j}\right)^{1+s}\right| \leq C_s\frac{|z-w_j|}{1-|\kappa|^2}.$$

Thus for $z \in T\left(w_j^\alpha\right)$,

$$\begin{aligned}
|\Phi(z) - \Phi(w_j)| &\leq \sum_{\kappa \in \left[o, w_j^\alpha\right]} h(\kappa) \left|\varphi_\kappa(z) - \varphi_\kappa(w_j)\right| + C \sum_{\kappa \notin [o,z]} h(\kappa) \left|\varphi_\kappa(z)\right| \\
&\leq C_s \sum_{\kappa \in \left[o, w_j^\alpha\right]} h(\kappa) \frac{|z - w_j|}{1 - |\kappa|^2} + CII(z) \\
&\leq C_s Cap_{\mathcal{T}}(E, F),
\end{aligned}$$

since $h(\kappa) \leq C\ Cap_{\mathcal{T}}(E, F)$ and $\sum_{\kappa \in [o, w_j]} \frac{1}{1 - |\kappa|^2} \approx \frac{1}{1 - |w_j|^2}$. This proves the first line in (2.17).

Moreover, we note that for $s = 0$ and $\kappa \in [o, w_j]$,

$$\operatorname{Re} \varphi_\kappa(w_j) = \operatorname{Re} \frac{1 - |\kappa|^2}{1 - \overline{\kappa} w_j} = \operatorname{Re} \frac{1 - |\kappa|^2}{\left|1 - \overline{\kappa} w_j\right|^2} (1 - \kappa \overline{w_j}) \geq c > 0.$$

A similar result holds for $s > -1$ provided the Bergman tree $\mathcal{T}$ is constructed sufficiently thin depending on s. It then follows from $\sum_{\kappa \in [o, w_j]} h(\kappa) = 1$ that

$$\begin{aligned}
\operatorname{Re} \Phi(w_j) &= \sum_{\kappa \in [o, w_j]} h(\kappa) \operatorname{Re} \varphi_\kappa(w_j) + \sum_{\kappa \notin [o, w_j]} h(\kappa) \operatorname{Re} \varphi_\kappa(w_j) \\
&\geq c \sum_{\kappa \in [o, w_j]} h(\kappa) - C\ Cap_{\mathcal{T}}(E, F) \geq c' > 0.
\end{aligned}$$

We trivially have

$$|\Phi(w_j)| \leq I(z) + II(z) \leq C \sum_{\kappa \in [o, w_j]} h(\kappa) + C\ Cap_{\mathcal{T}}(E, F) \leq C,$$

and this completes the proof of (2.17).

Now we prove (2.18). From property 1 of Proposition 1 we obtain

$$\begin{aligned}
\int_{\mathbb{D}} |g(\zeta)|^2 dA &= \int_{\mathbb{D}} \left| \sum_{\kappa \in \mathcal{T}} h(\kappa) \frac{1}{|B_\kappa|} \frac{\left(1 - \overline{\zeta} \kappa\right)^{1+s}}{(1 - |\zeta|^2)^s} \chi_{B_\kappa}(\zeta) \right|^2 dA \\
&= \sum_{\kappa \in \mathcal{T}} |h(\kappa)|^2 \frac{1}{|B_\kappa|^2} \int_{B_\kappa} \frac{\left|1 - \overline{\zeta} \kappa\right|^{2+2s}}{(1 - |\zeta|^2)^{2s}} dA \\
&\approx \sum_{\kappa \in \mathcal{T}} |h(\kappa)|^2 \approx Cap_{\mathcal{T}}(E, F).
\end{aligned}$$

Finally (2.19) follows from (2.18) and Lemma 2.4 of [7]. $\square$

Corollary 5. *Let G be a finite union of arcs in the circle $\mathbb{T}$. Then*

$$Cap_{\mathbb{D}}(G) \approx Cap_{\mathcal{T}}(G), \tag{2.21}$$

where $Cap_{\mathbb{D}}$ denotes Stegenga's capacity on the circle $\mathbb{T}$.

Proof. To prove the inequality $\lessapprox$ in (2.21) we use Proposition 2 to obtain a test function for estimating the Stegenga capacity of G. We take $F = \{o\}$ and $E = G$

in Proposition 2. Let c, C be the constants in Proposition 2, and suppose that $Cap\,(E,F) \leq c/(3C)$. Set $\Psi\,(z) = \frac{3}{c}\,(\Phi\,(z) - \Phi\,(0))$. Then $\Psi\,(0) = 0$ and

$$\begin{aligned}\operatorname{Re}\Psi\,(z) &= \frac{3}{c}\left\{\operatorname{Re}\Phi\,(z) - \operatorname{Re}\Phi\,(0)\right\} \\ &\geq \frac{3}{c}\left\{c - 2C Cap_{\mathcal{T}}\,(E,F)\right\} \geq 1,\ z \in G.\end{aligned}$$

By definition (2.2) and (2.19) we have that for $G \subset \mathbb{T}$

$$\begin{aligned}Cap_{\mathbb{D}}(G) \leq \|\Psi\|_{\mathcal{D}}^2 &= \left(\frac{3}{c}\right)^2 \|\Phi\|_{\mathcal{D}}^2 \\ &\leq \left(\frac{3}{c}\right)^2 C\ Cap_{\mathcal{T}}\,(E,F) \leq C\ Cap_{\mathcal{T}} E \\ &= C\ Cap_{\mathcal{T}} G.\end{aligned}$$

To obtain the opposite inequality we use $\psi \in \mathcal{D}$, an extremal function for computing $Cap_{\mathbb{D}} G$. For $R > 0$, $z \in \mathbb{D}$ let $B(z,R)$ be the hyperbolic disk of radius R centered at z. Pick R large enough so that for all $\kappa \in \mathcal{T} \setminus \{o\}$ we have $B(\kappa, R) \supset \operatorname{convexhull}(B_\kappa \cup B_{\kappa^{-1}})$. Our candidate for estimating $Cap_{\mathcal{T}}$ is given by setting $h\,(o) = 0$ and

$$h\,(\kappa) = (1 - |\kappa|^2) \sup\left\{|\psi'(z)| : z \in B(\kappa, R)\right\};\ \kappa \in \mathcal{T} \setminus \{o\}.$$

We have the pointwise estimate

$$\begin{aligned}\operatorname{Re}\psi\,(\beta) \leq |\psi\,(\beta)| &\leq \sum_{\kappa \in [o,\beta]} \left|\psi\,(\kappa) - \psi\left(\kappa^{-1}\right)\right| \\ &\leq \sum_{\kappa \in [o,\beta]} \left|\kappa - \kappa^{-1}\right| \sup\left\{|\psi'(z)| : z \in \text{segment}\left(\kappa, \kappa^{-1}\right)\right\} \\ &\leq C \sum_{\kappa \in [o,\beta]} h(\kappa) = CIh(\beta).\end{aligned}$$

We have the norm estimate, with $z\,(\kappa)$ denoting the appropriate point in $B(\kappa, R)$,

$$\begin{aligned}\|h\|_{\ell^2(\mathcal{T})}^2 &= \sum_{\kappa \in \mathcal{T}} \left(1 - \left|\kappa^2\right|\right)^2 \left|\psi'\,(z\,(\kappa))\right|^2 \\ &\leq C \sum_{\kappa \in \mathcal{T}} \frac{\left(1 - \left|\kappa^2\right|\right)^2}{|B(\kappa, R)|} \int_{B(\kappa,R)} \left|\psi'\,(z)\right|^2 dA \\ &\leq C \sum_{\kappa \in \mathcal{T}} \int_{B(\kappa,R)} \left|\psi'\,(z)\right|^2 dA \\ &\leq C \int_{\mathbb{D}} \left|\psi'\,(z)\right|^2 dA \leq C\,\|\psi\|_{\mathcal{D}}^2.\end{aligned}$$

Here the first inequality uses he submean value property for the subharmonic function $|\psi'\,(z)|^2$, the second uses straightforward estimates for $|B(\kappa, R)|$, and the next estimate holds because the $B(\kappa, R)$ are approximately disjoint; $\sum \chi_{B(\kappa,R)}(z) \leq C$. Recalling definition (2.8) we find

$$Cap_{\mathcal{T}} G \leq C \left\|\frac{1}{c}\psi\right\|_{\mathcal{D}}^2 = \frac{C}{c^2} Cap_{\mathbb{D}} G.$$

□

Abbreviate $Cap_{\mathcal{T}_\theta}$ by Cap_θ, and let $T_\theta(E)$ be the $\mathcal{T}_\theta$-tent region corresponding to an open subset E of the circle $\mathbb{T}$. Recall that $T(E) = \bigcup_{I\subset E} T(I)$. Now define M by

$$M := \sup_{E \text{ open } \subset\mathbb{T}} \frac{\int_{\mathbb{T}} \mu_b\left(T_\theta(E)\right) d\theta}{\int_{\mathbb{T}} Cap_\theta(E)\, d\theta}. \tag{2.22}$$

Corollary 6. *We have* $\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx M$.

Proof. Using Corollary 5 and $T_\theta(E) \subset T(E)$, we have

$$\begin{aligned} M &\le C \sup_{E \text{ open } \subset\mathbb{T}} \frac{\int_{\mathbb{T}} \mu_b\left(T(E)\right) d\theta}{\int_{\mathbb{T}} Cap_{\mathbb{D}}(E)\, d\theta} \\ &= C \sup_{E \text{ open } \subset\mathbb{T}} \frac{\mu_b\left(T(E)\right)}{Cap_{\mathbb{D}}(E)} \approx \|\mu_b\|^2_{\mathcal{D}-Carleson}, \end{aligned}$$

where the final comparison is Stegenga's theorem. Conversely, one can verify using an argument in the style of the one in (2.25) below that for $0<\rho<1$,

$$\begin{aligned} \mu_b(E) &\le C \int_{\mathbb{T}} \mu_b\left(T_\theta\left(E^\rho_{\mathbb{D}}\right)\right) d\theta \\ &\le CM \int_{\mathbb{T}} Cap_\theta\left(E^\rho_{\mathbb{D}}\right) d\theta \\ &\approx CM Cap_{\mathbb{D}}\left(E^\rho_{\mathbb{D}}\right) \\ &\le CM Cap_{\mathbb{D}}(E). \end{aligned}$$

Here the third line uses (2.21) with $E^\rho_{\mathbb{D}}$ and $\mathcal{T}(\theta)$ in place of G and $\mathcal{T}$, and the final inequality follows from (2.4). Thus from Stegenga's theorem we obtain

$$\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx \sup_{E \text{ open } \subset\mathbb{T}} \frac{\mu_b(E)}{Cap_{\mathbb{D}}(E)} \le CM.$$

□

Given $0<\delta<1$, let G be an open set in $\mathbb{T}$ such that

$$\frac{\int_{\mathbb{T}} \mu_b\left(T_\theta(G)\right) d\theta}{\int_{\mathbb{T}} Cap_\theta(G)\, d\theta} \ge \delta M. \tag{2.23}$$

We need to know that $\mu_b(V^\beta_G \setminus V_G)$ is small compared to $\mu_b(V_G)$. This crucial step of the proof is where we use the asymptotic capacity estimate Lemma 4.

Proposition 3. *Given $\varepsilon>0$ we can choose $\delta=\delta(\varepsilon)<1$ in (2.23) and $\beta=\beta(\varepsilon)<1$ so that for any G satisfying (2.23), we have, with $V^\beta_G = G^\beta_{\mathbb{D}}$ and $V_G = G^1_{\mathbb{D}} = T(G)$,*

$$\mu_b(V^\beta_G \setminus V_G) \le \varepsilon \mu_b(V_G), \tag{2.24}$$

Proof. Let $G^\rho(\theta) = G^\rho_{\mathcal{T}_\theta}$. Lemma 4 shows that $Cap_\theta(G^\rho(\theta)) \le \rho^{-2} Cap_\theta(G)$, for $0\le\theta<2\pi$, $0<\rho<1$, and if we integrate on $\mathbb{T}$ we obtain

$$\int_{\mathbb{T}} Cap_\theta\left(G^\rho(\theta)\right) d\theta \le \rho^{-2} \int_{\mathbb{T}} Cap_\theta(G)\, d\theta.$$

From (2.22) and (2.23) we thus have

$$\begin{aligned}\int_{\mathbb{T}} \mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right)\right) d\theta &\leq M \int_{\mathbb{T}} Cap_\theta\left(G^\rho\left(\theta\right)\right) d\theta \\ &\leq M\rho^{-2} \int_{\mathbb{T}} Cap_\theta\left(G\right) d\theta \\ &\leq \frac{1}{\delta\rho^2} \int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta.\end{aligned}$$

It follows that

$$\begin{aligned}\int\limits_{\mathbb{T}} \mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right) \setminus T_\theta\left(G\right)\right) d\theta &= \int\limits_{\mathbb{T}} \mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right)\right) d\theta - \int\limits_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta \\ &\leq \left(\frac{1}{\delta\rho^2} - 1\right) \int\limits_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta.\end{aligned}$$

Now with $\eta = (\rho + 1)/2$,

$$\begin{aligned}\int\limits_{\mathbb{T}} \mu_b\left(T_\theta\left(G^\rho\left(\theta\right)\right) \setminus T_\theta\left(G\right)\right) d\theta &= \int\limits_{\mathbb{T}} \int\limits_{T_\theta(G^\rho(\theta))\setminus T_\theta(G)} d\mu_b\left(z\right) d\theta \\ &\geq \int_{\mathbb{T}} \int\limits_{T_\theta(G^\rho(\theta))\setminus T(G)} d\mu_b\left(z\right) d\theta \\ &\geq \int_{\mathbb{T}} \int\limits_{T_\theta(G^\rho(\theta))\setminus T(G)} d\mu_b\left(z\right) d\theta \end{aligned} \tag{2.25}$$

$$= \int\limits_{\mathbb{D}} \left\{ \frac{1}{2\pi} \int\limits_{\{\theta : z \in T_\theta(G^\rho(\theta))\setminus T(G)\}} d\theta \right\} d\mu_b\left(z\right) \geq \frac{1}{2} \int_{T\left(G_{\mathbb{D}}^\eta\right)\setminus T(G)} d\mu_b\left(z\right),$$

since every $z \in T\left(G_{\mathbb{D}}^\eta\right)$ lies in $T_\theta\left(G^\rho\left(\theta\right)\right)$ for at least half of the θ's in $[0, 2\pi)$. Here we may assume that the components of $G_{\mathbb{D}}^\rho$ have small length since otherwise we trivially have $\int_{\mathbb{T}} Cap_{\mathcal{T}(\theta)}\left(G\right) d\theta \geq c > 0$. We continue with

$$M \leq \frac{1}{c} \int d\mu_b \leq \frac{1}{c} \left\|b\right\|_{\mathcal{D}}^2 \leq \frac{C}{c} \left\|T_b\right\|^2. \tag{2.26}$$

Combining the above inequalities, using $\rho = 2\eta - 1$, $1/2 \leq \rho < 1$, and choosing $\delta = \eta$, we obtain

$$\begin{aligned}\mu_b\left(T\left(G_{\mathbb{D}}^\eta\right) \setminus T\left(G\right)\right) &\leq 2\left(\frac{1}{\delta\rho^2} - 1\right) \int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta \\ &= 2\left(\frac{1}{\eta\left(2\eta - 1\right)^2} - 1\right) \int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta \\ &\leq C\left(1 - \eta\right) \int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta,\end{aligned}$$

for $3/4 \leq \eta < 1$. Recalling that $V_G^\eta = T\left(G_{\mathbb{D}}^\eta\right)$ and that for all θ we have $T_\theta\left(G\right) \subset T\left(G\right) = V_G$ this becomes

$$\mu_b\left(V_G^\eta \setminus V_G\right) \leq C\left(1 - \eta\right) \int_{\mathbb{T}} \mu_b\left(T_\theta\left(G\right)\right) d\theta \leq C\left(1 - \eta\right) \mu_b\left(V_G\right), \ \frac{3}{4} \leq \eta < 1,$$

Hence given $\varepsilon > 0$ it is possible to select δ and β so that (2.24) holds. $\square$

2.6. **Schur Estimates and a Bilinear Operator on Trees.** We begin with a bilinear version of Schur's well known theorem.

Proposition 4. *Let (X, μ), (Y, ν) and (Z, ω) be measure spaces and $H(x, y, z)$ be a nonnegative measurable function on $X \times Y \times Z$. Define, initially for nonnegative functions f, g,*

$$T(f, g)(x) = \int_{Y \times Z} H(x, y, z) f(y) \, d\nu(y) \, g(z) \, d\omega(z), \;\; x \in X,$$

For $1 < p < \infty$, suppose there are positive functions h, k and m on X, Y and Z respectively such that

$$\int_{Y \times Z} H(x, y, z) k(y)^{p'} m(z)^{p'} \, d\nu(y) \, d\omega(z) \leq (Ah(x))^{p'},$$

for μ-a.e. $x \in X$, and

$$\int_X H(x, y, z) h(x)^p \, d\mu(x) \leq (Bk(y) m(z))^p,$$

for $\nu \times \omega$-a.e. $(y, z) \in Y \times Z$. Then T is bounded from $L^p(\nu) \times L^p(\omega)$ to $L^p(\mu)$ and $\|T\|_{operator} \leq AB$.

Proof. We have

$$\begin{aligned}
&\int_X |Tf(x)|^p \, d\mu(x) \\
&\leq \int_X \left(\int_{Y \times Z} H(x, y, z) k(y)^{p'} m(z)^{p'} \, d\nu(y) \, d\omega(z) \right)^{p/p'} \\
&\times \left(\int_{Y \times Z} H(x, y, z) \left(\frac{f(y)}{k(y)} \right)^p d\nu(y) \left(\frac{g(z)}{m(z)} \right)^p d\omega(z) \right) d\mu(x) \\
&\leq A^p \int_{Y \times Z} \left(\int_X H(x, y, z) h(x)^p \, d\mu(x) \right) \left(\frac{f(y)}{k(y)} \right)^p d\nu(y) \left(\frac{g(z)}{m(z)} \right)^p d\omega(z) \\
&\leq A^p B^p \int_{Y \times Z} k(y)^p m(z)^p \left(\frac{f(y)}{k(y)} \right)^p d\nu(y) \left(\frac{g(z)}{m(z)} \right)^p d\omega(z) \\
&= (AB)^p \int_Y f(y)^p \, d\nu(y) \int_Z g(z)^p \, d\omega(z).
\end{aligned}$$

$\square$

This proposition can be used along with the estimates

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - \overline{w} z|^{2+t+c}} dw \approx \begin{cases} C_t & \text{if} \quad c < 0, \; t > -1 \\ -C_t \log(1 - |z|^2) & \text{if} \quad c = 0, \; t > -1 \\ C_t (1 - |z|^2)^{-c} & \text{if} \quad c > 0, \; t > -1 \end{cases}, . \tag{2.27}$$

to prove a corollary we will use later [18, Thm 2.10].

Corollary 7. *Define*

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{(1-\overline{w}z)^{2+a+b}} f(w)\,dw,$$
$$Sf(z) = (1-|z|^2)^a \int_{\mathbb{D}} \frac{(1-|w|^2)^b}{|1-\overline{w}z|^{2+a+b}} f(w)\,dw.$$

Suppose $t \in \mathbb{R}$ and $1 \le p < \infty$. Then T is bounded on $L^p\left(\mathbb{D},(1-|z|^2)^t dA\right)$ if and only if S is bounded on $L^p\left(\mathbb{D},(1-|z|^2)^t dA\right)$ if and only if

$$-pa < t+1 < p(b+1). \tag{2.28}$$

We now use Proposition 4 to show that if $\mathcal{A}$, $\mathcal{B} \subset \mathcal{T}$ are well separated then a certain bilinear operator mapping on $\ell^2(\mathcal{A}) \times \ell^2(\mathcal{B})$ maps boundedly into $L^2(\mathbb{D})$.

Lemma 6. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are subsets of T, $h \in \ell^2(\mathcal{A})$ and $k \in \ell^2(\mathcal{B})$, and $1/2 < \alpha < 1$. Suppose further that $\mathcal{A}$ and $\mathcal{B}$ satisfy the separation condition: $\forall \kappa \in \mathcal{A}$, $\gamma \in \mathcal{B}$ we have*

$$|\kappa - \gamma| \ge (1-|\gamma|^2)^\alpha. \tag{2.29}$$

Then the bilinear map of (h,k) to functions on the disk given by

$$T(h,b)(z) = \left(\sum_{\kappa\in\mathcal{A}} h(\kappa) \frac{(1-|\kappa|^2)^{1+s}}{|1-\overline{\kappa}z|^{2+s}}\right)\left(\sum_{\gamma\in\mathcal{B}} b(\gamma) \frac{(1-|\gamma|^2)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}\right)$$

is bounded from $\ell^2(\mathcal{A}) \times \ell^2(\mathcal{B})$ to $L^2(\mathbb{D})$.

Remark 1. *For $h \in \ell^2(\mathcal{A})$ and $b \in \ell^2(\mathcal{B})$ set*

$$H(z) = \sum_{\kappa\in\mathcal{A}} h(\kappa) \frac{(1-|\kappa|^2)^{1+s}}{(1-\overline{\kappa}z)^{2+s}},\ B(z) = \sum_{\gamma\in\mathcal{B}} b(\gamma) \frac{(1-|\gamma|^2)^{1+s}}{(1-\overline{\gamma}z)^{1+s}}.$$

By [18, Thm 2.30] *$H \in L^2(\mathbb{D})$ and $B \in \mathcal{D}$. There are unbounded functions in $\mathcal{D}$ hence these facts do not insure show $HB \in L^2(\mathbb{D})$. The lemma shows that if $\mathcal{A}$ and $\mathcal{B}$ are separated then $HB \in L^2(\mathbb{D})$.*

Proof. We will verify the hypotheses of the previous proposition. The kernel function here is

$$H(z,\kappa,\gamma) = \frac{(1-|\kappa|^2)^{1+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{(1-|\gamma|^2)^{1+s}}{|1-\overline{\gamma}z|^{1+s}},\ z \in \mathbb{D}, \kappa \in \mathcal{A}, \gamma \in \mathcal{B},$$

with Lebesgue measure on $\mathbb{D}$, and counting measure on $\mathcal{A}$ and $\mathcal{B}$. We will take as Schur functions

$$h(z) = (1-|z|^2)^{-1/4},\ k(\kappa) = (1-|\kappa|^2)^{1/4} \text{ and } m(\gamma) = (1-|\gamma|^2)^{\varepsilon/2},$$

on $\mathbb{D}$, $\mathcal{A}$ and $\mathcal{B}$ respectively, where $\varepsilon = \varepsilon(\alpha,s) > 0$ will be chosen sufficiently small later. We must then verify

$$\sum_{\kappa\in\mathcal{A}} \sum_{\gamma\in\mathcal{B}} \frac{(1-|\kappa|^2)^{3/2+s}}{|1-\overline{\kappa}z|^{2+s}} \frac{(1-|\gamma|^2)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}} \le A^2(1-|z|^2)^{-1/2}, \tag{2.30}$$

for $z\in\mathbb{D}$, and

$$\int_{\mathbb{D}}\frac{(1-|\kappa|^2)^{1+s}}{|1-\overline{\kappa}z|^{2+s}}\frac{(1-|\gamma|^2)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}(1-|z|^2)^{-1/2}dA\leq B^2(1-|\kappa|^2)^{1/2}(1-|\gamma|^2)^{\varepsilon}, \tag{2.31}$$

for $\kappa\in\mathcal{A}$ and $\gamma\in\mathcal{B}$. □

Lemma 7. *Proof.* To prove (2.30) we write

$$\sum_{\kappa\in\mathcal{A}}\sum_{\gamma\in\mathcal{B}}\frac{(1-|\kappa|^2)^{3/2+s}}{|1-\overline{\kappa}z|^{2+s}}\frac{(1-|\gamma|^2)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}= \left(\sum_{\kappa\in\mathcal{A}}\frac{(1-|\kappa|^2)^{3/2+s}}{|1-\overline{\kappa}z|^{2+s}}\right)\left(\sum_{\gamma\in\mathcal{B}}\frac{(1-|\gamma|^2)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}\right).$$

Then from (2.27) we obtain

$$\sum_{\kappa\in\mathcal{A}}\frac{(1-|\kappa|^2)^{3/2+s}}{|1-\overline{\kappa}z|^{2+s}}\leq C\int_{\mathbb{D}}\frac{(1-|w|^2)^{-1/2+s}}{|1-\overline{w}z|^{2+s}}dw\leq C(1-|z|^2)^{-1/2}$$

and

$$\sum_{\gamma\in\mathcal{B}}\frac{(1-|\gamma|^2)^{1+\varepsilon+s}}{|1-\overline{\gamma}z|^{1+s}}\leq C\int_{\zeta\in V_G}\frac{(1-|\zeta|^2)^{-1+\varepsilon+s}}{\left|1-\overline{\zeta}z\right|^{1+s}}dA\leq C,$$

which yields (2.30).

We now prove (2.31) We will make repeated use of (2.29) as well as its consequence via the triangle inequality: $\forall\kappa\in\mathcal{A}$, $\gamma\in\mathcal{B}$ $(1-|\kappa|^2)\leq C|\kappa-\gamma|$. We set $\kappa^*=\kappa/|\kappa|$, $\gamma^*=\gamma/|\gamma|$.

$$\begin{aligned}&\int_{\mathbb{D}}\frac{(1-|\kappa|^2)^{1+s}}{|1-\overline{\kappa}z|^{2+s}}\frac{(1-|\gamma|^2)^{1+s}}{|1-\overline{\gamma}z|^{1+s}}(1-|z|^2)^{-1/2}dA\\ &=\int_{|z-\gamma^*|\leq 1-|\gamma|^2}+\int_{1-|\gamma|^2\leq|z-\gamma^*|\leq\frac{1}{2}|\kappa-\gamma|}\\ &+\int_{|z-\kappa^*|\leq 1-|\kappa|^2}+\int_{1-|\kappa|^2\leq|z-\kappa^*|\leq\frac{1}{2}|\kappa-\gamma|}+\int_{|z-\gamma^*|,|z-\kappa^*|\geq|\kappa-\gamma|}...dA\\ &=I+II+III+IV+V.\end{aligned}$$

We have

$$\begin{aligned}I&\approx\frac{(1-|\kappa|^2)^{1+s}}{|\kappa-\gamma|^{2+s}}\int_{|z-\gamma^*|\leq 1-|\gamma|^2}(1-|z|^2)^{-1/2}dA\\ &\approx\frac{(1-|\kappa|^2)^{1+s}(1-|\gamma|^2)^{3/2}}{|\kappa-\gamma|^{2+s}}\leq C(1-|\kappa|^2)^{1/2}(1-|\gamma|^2)^{3(1-\alpha)/2}.\end{aligned}$$

Similarly we have

$$\begin{aligned} II &\approx \frac{(1-|\kappa|^2)^{1+s}(1-|\gamma|^2)^{1+s}}{|\kappa-\gamma|^{2+s}} \int\limits_{1-|\gamma|^2\le|z-\gamma^*|\le\frac{1}{2}|\kappa-\gamma|} \frac{(1-|z|^2)^{-1/2}}{|z-\gamma^*|^{1+s}} dA \\ &\approx \frac{(1-|\kappa|^2)^{1+s}(1-|\gamma|^2)^{1+s}}{|\kappa-\gamma|^{2+s}}(1-|\gamma|^2)^{1/2-s} \\ &= \frac{(1-|\kappa|^2)^{1+s}(1-|\gamma|^2)^{3/2}}{|\kappa-\gamma|^{2+s}} \le C(1-|\kappa|^2)^{1/2}(1-|\gamma|^2)^{3(1-\alpha)/2}. \end{aligned}$$

Continuing we obtain

$$III \approx \frac{(1-|\kappa|^2)^{1/2}(1-|\gamma|^2)^{1+s}}{|\kappa-\gamma|^{1+s}} \le C(1-|\kappa|^2)^{1/2}(1-|\gamma|^2)^{(1+s)(1-\alpha)},$$

and similarly,

$$IV \le C(1-|\kappa|^2)^{1/2}(1-|\gamma|^2)^{\varepsilon},$$

for some $\varepsilon > 0$. Finally

$$\begin{aligned} V &\approx \int\limits_{|z-\gamma^*|,|z-\kappa^*|\ge|\kappa-\gamma|} \frac{(1-|\kappa|^2)^{1+s}}{|z-\kappa^*|^{2+s}} \frac{(1-|\gamma|^2)^{1+s}}{|z-\gamma^*|^{1+s}}(1-|z|^2)^{-1/2} dA \\ &\approx \frac{(1-|\kappa|^2)^{1+s}(1-|\gamma|^2)^{1+s}}{|\kappa-\gamma|^{3/2+2s}} \\ &\le C(1-|\kappa|^2)^{1/2}(1-|\gamma|^2)^{(1+s)(1-\alpha)}. \end{aligned}$$

□

3. The Main Bilinear Estimate

To complete the proof we will show that μ_b is a $\mathcal{D}$-Carleson measure by verifying Stegenga's condition (2.3); that is, we will show that for any finite collection of disjoint arcs $\{I_j\}_{j=1}^N$ in the circle $\mathbb{T}$ we have

$$\mu_b\left(\bigcup_{j=1}^N T(I_j)\right) \le C\ Cap_{\mathbb{D}}\left(\bigcup_{j=1}^N I_j\right).$$

In fact we will see that it suffices to verify this for the sets $G = \cup_{j=1}^N I_j$ described in (2.23) that are almost extremal for (2.22). We will prove the inequality

$$\mu_b(V_G) \le C\,\|T_b\|^2\, Cap_{\mathbb{D}}(G). \tag{3.1}$$

Once we have this Corollary 5 yields

$$M = \frac{\int_{\mathbb{T}} \mu_b(T_\theta(G))\,d\theta}{\int_{\mathbb{T}} Cap_\theta(G)\,d\theta} \le \frac{\mu_b(V_G)}{\int_{\mathbb{T}} Cap_\theta(G)\,d\theta} \le C\,\|T_b\|^2.$$

By Corollary 6 $\|\mu_b\|^2_{\mathcal{D}-Carleson} \approx M$ which then completes the proof of Theorem 1.

We now turn to (3.1). Let $1/2 < \beta < \beta_1 < \gamma < \alpha < 1$ with additional constraints to be added later. Suppose G (2.23) with $\varepsilon > 0$ to be chosen below. We define in

succession the following regions in the disk,

$$\begin{aligned}
V_G &= T_{\mathcal{T}}(G),\\
V_G^{\alpha} &= G_{\mathbb{D}}^{\alpha},\\
V_G^{\gamma} &= \widehat{(V_G^{\alpha})_{\mathcal{T}}^{\gamma/\alpha}},\\
V_G^{\beta} &= (V_G^{\gamma})_{\mathbb{D}}^{\beta/\gamma}.
\end{aligned}$$

Thus V_G is the $\mathcal{T}$-tent associated with G, V_G^{α} is a disk blowup of G, V_G^{γ} is a $\mathcal{T}$-capacitary blowup of V_G^{α}, and V_G^{β} is a disk blowup of V_G^{γ}. Using the natural bijections described earlier, we write

$$V_G = \{w_k\}_k \text{ and } V_G^{\alpha} = \{w_k^{\alpha}\}_k \text{ and } V_G^{\gamma} = \{w_k^{\gamma}\}_k \text{ and } V_G^{\beta} = \{w_k^{\beta}\}_k, \tag{3.2}$$

with $w_k, w_k^{\alpha}, w_k^{\gamma}, w_k^{\beta} \in \mathcal{T}$. Following earlier notation we write $E = V_G^{\alpha}$ and $F = V_G^{\gamma}$.

We proceed by estimating $T_b(f,g)$ for well chosen f and g in $\mathcal{D}$. Let Φ be as in (2.12); we then have the estimates in Proposition 2 and Corollary 4. Set $g = \Phi^2$; in particular note that g is approximately equal to χ_{V_G}. The function f will be, approximately, $b'\chi_{V_G}$;

$$f(z) = \Gamma_s\left(\frac{1}{(1+s)\overline{\zeta}}\chi_{V_G} b'(\zeta)\right)(z) = \int_{V_G} \frac{b'(\zeta)(1-|\zeta|^2)^s}{\left(1-\overline{\zeta}z\right)^{1+s}}\frac{dA}{(1+s)\overline{\zeta}}. \tag{3.3}$$

We now analyze $T_b(f,g)$. From (3.3) and (2.15) we have

$$\begin{aligned}
f'(z) &= \int_{V_G} \frac{b'(\zeta)(1-|\zeta|^2)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA\\
&= b'(z) - \int_{\mathbb{D}\setminus V_G} \frac{b'(\zeta)(1-|\zeta|^2)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA\\
&= b'(z) + \Lambda b'(z),
\end{aligned}$$

where the last term is defined by

$$\Lambda b'(z) = -\int_{\mathbb{D}\setminus V_G} \frac{b'(\zeta)(1-|\zeta|)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA. \tag{3.4}$$

We have

$$\begin{aligned}
T_b(f,g) &= \left(f\Phi^2\overline{b}\right)(0) + \int_{\mathbb{D}} \{f'(z)\Phi(z) + 2f(z)\Phi'(z)\}\Phi(z)\overline{b'(z)}dA \\
&= \left(f\Phi^2\overline{b}\right)(0) + \int_{\mathbb{D}} |b'(z)|^2 \Phi(z)^2 dA\\
&+ 2\int_{\mathbb{D}} \Phi(z)\Phi'(z) f(z)\overline{b'(z)}dA + \int_{\mathbb{D}} \Lambda b'(z)\overline{b'(z)}\Phi(z)^2 dA\\
&= (1) + (2) + (3) + (4).
\end{aligned} \tag{3.5}$$

Now we write

$$(2) = \int_{\mathbb{D}} |b'(z)|^2 \Phi(z)^2 \, dA = \left\{ \int_{V_G} + \int_{V_G^\beta \setminus V_G} + \int_{\mathbb{D} \setminus V_G^\beta} \right\} |b'(z)|^2 \Phi(z)^2 \, dA = (2_A) + (2_B) + (2_C). \tag{3.6}$$

The main term is (2_A). By (2.17) and (2.1) it satisfies

$$(2_A) = \mu_b(V_G) + \int_{V_G} |b'(z)|^2 (\Phi(z)^2 - 1) dA = \mu_b(V_G) + O(\|T_b\|^2 \, Cap_{\mathcal{T}}(E,F)), \tag{3.7}$$

Rearranging this and using (3.5) and (3.6) we find

$$\mu_b(V_G) \leq C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F) + |T_b(f,g)| + |(1)| + (2_B) + (2_C) + |(3)| + |(4)| \tag{3.8}$$

Using the boundedness of T_b and Corollary 4 we have

$$|T_b(f,g)| = \left|T_b(f,\Phi^2)\right| = |T_b(f\Phi,\Phi)| \leq \|T_b\| \, \|f\Phi\|_{\mathcal{D}} \, \|\Phi\|_{\mathcal{D}} \leq C \|T_b\| \, \|f\Phi\|_{\mathcal{D}} \sqrt{Cap_{\mathcal{T}}(E,F)}. \tag{3.9}$$

For (1) we use the elementary estimate

$$|(1)| \leq C \|b\|_{\mathcal{D}}^2 \, Cap_{\mathcal{T}}(E,F) \leq C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F).$$

For (2_B) we use (2.24) to obtain

$$(2_B) \leq C\mu_b\left(V_G^\beta \setminus V_G\right) \leq C\varepsilon \mu_b(V_G). \tag{3.10}$$

Using (2.17) once more, we see that (2_C) satisfies

$$(2_C) \leq \int_{\mathbb{D} \setminus V_G^\beta} |b'(z)|^2 \left(C_{\alpha,\beta,\rho} Cap_{\mathcal{T}}(E,F)\right)^2 dA \leq C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F). \tag{3.11}$$

Putting these estimates into (3.8) we obtain

$$\mu_b(V_G) \leq C \left(\|T_b\|^2 \, Cap_{\mathcal{T}}(E,F) + \|T_b\| \, \|f\Phi\|_{\mathcal{D}} \sqrt{Cap_{\mathcal{T}}(E,F)} + |(3)| + |(4)| \right). \tag{3.12}$$

For small positive ε we estimate (3) using Cauchy-Schwarz as follows:

$$|(3)| \leq 2 \int_{\mathbb{D}} |\Phi(z) b'(z)| \, |\Phi'(z) f(z)| \, dA \leq \varepsilon \int_{\mathbb{D}} |\Phi(z) b'(z)|^2 \, dA + \frac{C}{\varepsilon} \int_{\mathbb{D}} |\Phi'(z) f(z)|^2 \, dA = (3_A) + (3_B).$$

Using the decomposition and the argument surrounding term (2) we obtain

$$(3_A) \leq \varepsilon \left\{ \int_{V_G} + \int_{V_G^\beta \setminus V_G} + \int_{\mathbb{D} \setminus V_G^\beta} \right\} |\Phi(z) b'(z)|^2 \, dA \leq C\varepsilon \left(\mu_b(V_G) + C \|T_b\|^2 \, Cap_{\mathcal{T}}(E,F) \right). \tag{3.13}$$

To estimate term (3_B) we use

$$\begin{aligned}
|f(z)| &\leq \left|\Gamma_s\left(\frac{1}{(1+s)\overline{\zeta}}\chi_{V_G}b'(\zeta)\right)(z)\right| \\
&\leq \int_{V_G}\frac{\left(1-|\zeta|^2\right)^s}{\left|1-\overline{\zeta}z\right|^{1+s}}\left|b'(\zeta)\right|dA \\
&\approx \sum_{\gamma\in\mathcal{T}\cap V_G}\frac{(1-|\gamma|^2)^{1+s}}{\left|1-\overline{\gamma}z\right|^{1+s}}\int_{B_\gamma}\left|b'(\zeta)\right|\left(1-|\zeta|^2\right)d\lambda(\zeta) \\
&= \sum_{\gamma\in\mathcal{T}\cap V_G}\frac{(1-|\gamma|^2)^{1+s}}{\left|1-\overline{\gamma}z\right|^{1+s}}b(\gamma),
\end{aligned}$$

where

$$\sum_{\gamma\in\mathcal{T}\cap V_G}b(\gamma)^2\approx\sum_{\gamma\in\mathcal{T}\cap V_G}\int_{B_\gamma}\left|b'(\zeta)\right|^2(1-|\zeta|^2)^2d\lambda(\zeta)=\int_{V_G}\left|b'(\zeta)\right|^2dA.$$

We now use the separation of $\mathbb{D}\setminus V_G^\alpha$ and V_G. The facts that $\mathcal{A}=supp\,(h)\subset\mathbb{D}\setminus V_G^\alpha$ and $\mathcal{B}=\mathcal{T}\cap V_G\subset V_G$, together with Lemma 1, insure that (2.29) is satisfied and hence we can use Lemma 6 and the representation of Φ in (2.12) to continue with

$$(3_B)=\int_{\mathbb{D}}\left|\Phi'(z)f(z)\right|^2dA\leq C\left(\sum_{\kappa\in\mathcal{A}}h(\kappa)^2\right)\left(\sum_{\gamma\in\mathcal{B}}b(\gamma)^2\right).$$

We also have from (2.1) and Corollary 4 that

$$\left(\sum_{\kappa\in\mathcal{A}}h(\kappa)^2\right)\left(\sum_{\gamma\in\mathcal{B}}b(\gamma)^2\right)\leq CCap_{\mathcal{T}}(E,F)\left\|T_b\right\|^2.$$

Altogether we then have

$$(3_B)\leq C\ Cap_{\mathcal{T}}(E,F)\left\|T_b\right\|^2, \tag{3.14}$$

and thus also

$$|(3)|\leq\varepsilon\mu_b(V_G)+C\left\|T_b\right\|^2Cap_{\mathcal{T}}(E,F). \tag{3.15}$$

We begin our estimate of term (4) by

$$\begin{aligned}
|(4)| &= \left|\int_{\mathbb{D}}\Lambda b'(z)\overline{b'(z)}\Phi(z)^2\,dA\right| \qquad (3.16)\\
&\leq \sqrt{\int_{\mathbb{D}}\left|b'(z)\Phi(z)\right|^2dA}\sqrt{\int_{\mathbb{D}}\left|\Lambda b'(z)\Phi(z)\right|^2dA}.
\end{aligned}$$

where the first factor is $\sqrt{(3_A)/\varepsilon}$. We claim the following estimate for the second factor $\sqrt{(4_A)}:=\left\|\Phi\Lambda b'\right\|_{L^2(\mathbb{D})}$:

Lemma 8.

$$(4_A)=\int_{\mathbb{D}}\left|\Phi(z)\Lambda b'(z)\right|^2dA\leq C\mu_b\left(V_G^\beta\setminus V_G\right)+C\left\|T_b\right\|^2Cap_{\mathcal{T}}(E,F) \tag{3.17}$$

Proof. From (3.4) we obtain

$$\begin{aligned}
(4_A) &= \int_{\mathbb{D}} \left|\Phi(z)\right|^2 \left| \left\{ \int_{V_G^\beta \setminus V_G} + \int_{\mathbb{D}\setminus V_G^\beta} \right\} \frac{b'(\zeta)(1-|\zeta|)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA \right|^2 dA \\
&\leq C \int_{\mathbb{D}} \left|\Phi(z)\right|^2 \left(\int_{V_G^\beta \setminus V_G} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} dA \right)^2 dA \\
&\qquad + C \int_{\mathbb{D}} \left|\Phi(z)\right|^2 \left| \int_{\mathbb{D}\setminus V_G^\beta} \frac{b'(\zeta)(1-|\zeta|)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA \right|^2 dA \\
&= (4_{AA}) + (4_{AB}).
\end{aligned}$$

Corollary 7 shows that

$$\begin{aligned}
|(4_{AA})| &\leq \int_{\mathbb{D}} \left(\int_{V_G^\beta \setminus V_G} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} dA \right)^2 dA \\
&\leq C \int_{V_G^\beta \setminus V_G} \left|b'(\zeta)\right|^2 dA = C\mu_b\left(V_G^\beta \setminus V_G\right).
\end{aligned}$$

We write the second integral as

$$\begin{aligned}
(4_{AB}) &= \left\{ \int_{V_G^\gamma} + \int_{\mathbb{D}\setminus V_G^\gamma} \right\} \left|\Phi(z)\right|^2 \left| \int_{\mathbb{D}\setminus V_G^\beta} \frac{b'(\zeta)(1-|\zeta|)^s}{\left(1-\overline{\zeta}z\right)^{2+s}} dA \right|^2 dA \\
&= (4_{ABA}) + (4_{ABB}),
\end{aligned}$$

where by Corollary 7 again,

$$\begin{aligned}
(4_{ABB}) &\leq C\; Cap_{\mathcal{T}}(E,F)^2 \int_{\mathbb{D}} \left|b'(\zeta)\right|^2 dA \\
&\leq C \left\|T_b\right\|^2 Cap_{\mathcal{T}}(E,F)^2 \\
&\leq C \left\|T_b\right\|^2 Cap_{\mathcal{T}}(E,F).
\end{aligned}$$

Finally, with $\beta < \beta_1 < \gamma < \alpha < 1$, Corollary 7 shows that the term (4_{ABA}) satisfies the following estimate. Recall that $V_G^\gamma = \cup J_k^\gamma$ and $w_j^\gamma = z(J_k^\gamma)$. We set $A_\ell = \left\{k : J_k^\gamma \subset J_\ell^{\beta_1}\right\}$ and define $\ell(k)$ by the condition $k \in A_{\ell(k)}$. From Lemma 1 we have that, with $\rho = \beta_1/\gamma$, sidelength$(J_k^\gamma) \leq$ sidelength$(J_\ell^{\beta_1})^{1/\rho}$. Hence

$$\begin{aligned}
(4_{ABA}) &\leq C \int_{V_G^\gamma} \left(\int_{\mathbb{D}\setminus V_G^\beta} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}} d\zeta \right)^2 dA \\
&\approx C \sum_k \int_{J_k^\gamma} |J_k^\gamma| \left(\int_{\mathbb{D}\setminus V_G^\beta} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}w_k^\gamma\right|^{2+s}} d\zeta \right)^2 dA \\
&= C \sum_k \frac{|J_k^\gamma|}{\left|J_{\ell(k)}^{\beta_1}\right|} \left|J_{\ell(k)}^{\beta_1}\right| \left(\int_{\mathbb{D}\setminus V_G^\beta} \frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}w_k^\gamma\right|^{2+s}} d\zeta \right)^2
\end{aligned}$$

$$\begin{aligned}
&\approx C\sum_{\ell}\frac{\sum_{k\in A_\ell}|J_k^\gamma|}{\left|J_\ell^{\beta_1}\right|}\int_{J_\ell^{\beta_1}}\left(\int_{\mathbb{D}\setminus V_G^\beta}\frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}}d\zeta\right)^2 dA\\
&\leq C\left|V_G^{\beta_1}\right|^{\varepsilon(\gamma-\beta_1)}\int_{V_G^{\beta_1}}\left(\int_{\mathbb{D}\setminus V_G^\beta}\frac{|b'(\zeta)|(1-|\zeta|)^s}{\left|1-\overline{\zeta}z\right|^{2+s}}d\zeta\right)^2 dA\\
&\leq C\left|V_G^{\beta_1}\right|^{\varepsilon(\gamma-\beta_1)}\|b\|_{\mathcal{D}}^2\leq C\|T_b\|^2\, Cap_{\mathcal{T}}(E,F).
\end{aligned}$$

We continue from (3.16). We know that $|(4)|\leq\sqrt{(3_A)/\varepsilon}\sqrt{(4_A)}$ We estimate (3_A) using (3.13) and (4_A) using (3.17). After that we continue by using (2.24);

$$\begin{aligned}
|(4)|&\leq\sqrt{C\mu_b(V_G)+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)}\\
&\quad\times\sqrt{C\mu_b\left(V_G^\beta\setminus V_G\right)+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)}\\
&\leq\sqrt{C\mu_b(V_G)+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)}\\
&\quad\times\sqrt{\varepsilon\mu_b(V_G)+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)}\\
&\leq\sqrt{\varepsilon}\mu_b(V_G)+C\sqrt{\mu_b(V_G)}\sqrt{\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)}\\
&\quad+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F).
\end{aligned}\tag{3.18}$$

Now, recalling that $f'=b'+\Lambda b'$,

$$\begin{aligned}
\|\Phi f\|_{\mathcal{D}}^2&\leq C\int|\Phi'(z)f(z)|^2\,dA+C\int|\Phi(z)(b'(z)+\Lambda b'(z))|^2\,dA\\
&\leq C(3_B)+C\frac{1}{\varepsilon}(3_A)+C(4_A).\\
&\leq C\mu_b(V_G)+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F),
\end{aligned}\tag{3.19}$$

by (3.17) and the estimates (3.13) and (3.14) for (3_A) and (3_B). □

Using Proposition 3 and the estimates (3.15), (3.18) and (3.19) in (3.12) we obtain

$$\begin{aligned}
\mu_b(V_G)&\leq\sqrt{\varepsilon}\mu_b(V_G)+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)\\
&\quad+C\sqrt{\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)}\sqrt{\mu_b(V_G)}\\
&\leq\sqrt{\varepsilon}\mu_b(V_G)+C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F).
\end{aligned}$$

We absorb the first term into the right side. Now using Lemma 3, Lemma 4 again, and Corollary 5 we obtain

$$Cap_{\mathcal{T}}(E,F)\leq C\,Cap_{\mathbb{D}}G.$$

Finally we have

$$\mu_b(V_G)\leq C\|T_b\|^2\,Cap_{\mathcal{T}}(E,F)\leq C\|T_b\|^2\,Cap_{\mathbb{D}}G,$$

which is (3.1).

4. Appendix on Tree Extremals

Let E be a stopping time in $\mathcal{T}$. Recall that

$$Cap_{\mathcal{T}}(E) = \inf\{\|h\|_{\ell^2}^2 : \; Ih \geq 1 \text{ on } E\}. \tag{4.1}$$

We call functions which can be used in computing the infimum *admissible.*

Much of the following proposition as well as Proposition 1 could be extracted from general capacity theory such as presented in, for instance, [1]. Statement (3) is the discrete analog of the fact that continuous capacity can be interpreted as the derivative at infinity of a Green function.

Proposition 5. *Suppose $E \subset \mathcal{T}$ is given.*

(1) *There is a function h such that the infimum in the definition of $Cap_{\mathcal{T}}(E)$ is achieved.*
(2) *If $x \notin E$,*

$$h(x) = h(x_+) + h(x_-). \tag{4.2}$$

(3) $h(o) = \|h\|_{\ell^2}^2$.
(4) *h is strictly positive on $\mathcal{G}(o, E)$ and zero elsewhere.*
(5) $Ih|_E = 1$.

Proof. Consider first the case when E is a finite subset of $\mathcal{T}$. Multiplying an admissible function by the characteristic function of $\mathcal{G}(o, E)$ leaves it admissible and reduces the ℓ^2 norm. Hence we need only consider functions supported on the finite set of vertices in $\mathcal{G}(o, E)$. In that context it is easy to see that an extremal exists, call it h. Now consider (2). Suppose $x \in \mathcal{T} \setminus E$ and consider the competing function h^* which takes the same values as h except possible at x, x_+, and x_- and whose values at those points are determined by

(1). $h^*(x) + h^*(x_+) = h(x) + h(x_+)$ and $h^*(x) + h^*(x_-) = h(x) + h(x_-)$
(2). $h^*(x)^2 + h^*(x_+)^2 + h^*(x_-)^2$ is minimal subject to (1).

Then h^* is admissible, $\|h^*\|_{\ell^2}^2 \leq \|h\|_{\ell^2}^2$, and, doing the calculus problem, h^* satisfies (4.2). Hence h must satisfy (4.2).

If $h(x) < 0$ at some point, replacing its value by zero leaves the function admissible while reducing the ℓ^2 norm, hence $h \geq 0$. To complete the proof of (4) we must show that we cannot have an $x \in \mathcal{G}(o, E)$ at which $h(x) = 0$. Suppose we had such a point. By (4.2) and the fact that $h \geq 0$, we have $h \equiv 0$ on $S_{\mathcal{T}}(x)$. Hence by admissibility $Ih(x^{-1}) \geq 1$. Let $y \neq x$ be the point such that $x^{-1} = y^{-1}$. If $h(y) > 0$ then setting $h(y) = 0$ we would decrease the ℓ^2 norm while keeping the function admissible. Thus $h(y) = 0$ and, by (4.2), $h(x^{-1}) = 0$. Continuing in this way we find that $h \equiv 0$ an the geodesic from o to some $e \in E$, an impossibility for an admissible function. Item (5) is a consequence of this. If $Ih(e) > 1$ for some $e \in E$ and $h(e) > 0$ then we could decrease $h(e)$ slightly, reducing the norm of h and still have h admissible; contradicting the supposition that h is extremal.

It remains to show (3) and we do that by induction on the size of E. If $E = \{e\}$ is a single point having distance $d - 1 \geq 0$ from o then the extremal is $h \equiv 1/d$ on $[o, e]$ and $\|h\|_{\ell^2}^2 = d(1/d)^2 = h(o)$. Given E with more than one point, let z be the uniquely determined branching point in $\mathcal{G}(o, E)$ having the least distance from the root. Consider the rooted trees $\mathcal{T}_\pm = S(z_\pm)$ with roots $z_\pm$. Set $E_\pm = E \cap \mathcal{T}_\pm$ and let $h_\pm$ be the extremal functions for the computation of $Cap_{\mathcal{T}_\pm}(E_\pm)$. By induction,

we have that $\|h_\pm\|^2_{\ell^2} = h_\pm(z_\pm)$. From properties (1)-(5) satisfied by the extremal functions h, h_+ and h it is easy to see that

$$h(x) = \begin{cases} (1 - Ih(z))h_\pm(x) \text{ if } x \in \mathcal{G}(z_\pm) \\ h(o) \text{ if } x \in [o, z] \\ 0 \text{ otherwise.} \end{cases}$$

In particular, $Ih(z) = dh(o)$ if there are d points in $[o, z]$, so that

$$h(o) = h(z) = h(z_+) + h(z_-) = \frac{h_+(z_+) + h_-(z_-)}{1 - Ih(z)} = \frac{h_+(z_+) + h_-(z_-)}{1 - dh(o)}. \tag{4.3}$$

Rescaling and using the induction hypothesis,

$$\begin{aligned} \|h\|^2_{\ell^2} &= \left(\|h_+\|^2_{\ell^2} + \|h_-\|^2_{\ell^2}\right)(1 - dh(o))^2 + dh(o)^2 \\ &= (h_+(z_+) + h_-(z_-))(1 - dh(o))^2 + dh(o)^2 \\ &= \frac{h(z_+) + h(z_-)}{1 - dh(o)}(1 - dh(o))^2 + dh(o)^2 \\ &= \frac{h(z)}{1 - dh(o)}(1 - dh(o))^2 + dh(o)^2 \\ &= \frac{h(o)}{1 - dh(o)}(1 - dh(o))^2 + dh(o)^2 \\ &= h(o). \end{aligned}$$

We note in passing that, by (3), formula (4.3) gives a recursive formula for computing tree capacities.

Suppose now that E is infinite. Select a sequence of finite sets $E_n = \{e_1, \dots, e_n\}$ such that $E_n \nearrow E$. Let h_n be the corresponding extremal functions and $H_n = Ih_n$. We claim that the sequence H_n increases, in the sense specified below. Let $K = H_n - H_{n-1} = I(h_n - h_{n-1}) = Ik_n$. By (4.2), the function K satisfies the mean value property on $\mathcal{G}(o, E_n) \setminus (\{o\} \cup E_n)$:

$$K(x) = \frac{1}{3}[K(x_+) + K(x_-) + K(x^{-1})], \text{ if } x \in \mathcal{G}(o, E_n) \setminus (\{o\} \cup E_n).$$

Moreover, K vanishes on $\{o\} \cup E_{n-1}$ and it is positive at e_n, since $H_{n-1}(e_n) \le 1 = H_n(e_n)$, by (3) and (4). By the maximum principle (an easy consequence of the mean value property), $K_n \ge 0$ in $\mathcal{G}(o, E_n)$. Hence, the limit $Ih = H = \lim_n H_n$ exists in $\mathcal{G}(o, E)$ and it is finite because each H_n is bounded above by 1. Since $h(x) = H(x) - H(x^{-1}) = \lim h_n(x)$, h is admissible for E and it satisfies (3), (4) and (5).

Also, $h_n \to h$ as $n \to \infty$, pointwise, and $\|h_n\|^2_{\ell^2} = h_n(o) \to h(o)$, by dominated convergence, hence,

$$h(o) = \lim_{n\to\infty} \|h_n\|^2_{\ell^2} = \|h\|^2_{\ell^2},$$

which is (3) for h.

It remains to prove that h is extremal. Suppose k is another admissible function for E, and let k_n be its restriction to $\mathcal{G}(o, E_n)$, which is clearly admissible for E_n. By the extremal character of the functions h_n, we have

$$\|k\|^2_{\ell^2} = \lim_{n\to\infty} \|k_n\|^2_{\ell^2} \le \lim_{n\to\infty} \|h_n\|^2_{\ell^2} = \lim_{n\to\infty} h_n(o) = h(o) = \|h\|^2_{\ell^2},$$

hence, h is extremal among the admissible functions for E. $\square$

Proof of Proposition 1. Consider each $e \in E$ as the root of the tree $\mathcal{T}_e = S(e)$. Set $F_e = F \cap S(e)$ and let h_e be the extremal function (from the previous proposition) for computing $Cap_{\mathcal{T}_e}(F_e)$. Using the previous proposition it is straightforward to check that $h = \sum h_e$ is the required extremal function and has the required properties. $\square$

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